Enumerative Geometry and Hilbert Schemes of Curves in $\mathbb{P}^3$

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The usual strategy to solve enumerative problems is:

First: to find a suitable parameter space.

Second: to express the number you are looking for as the degree of a class in the Chow ring of your parameter space.

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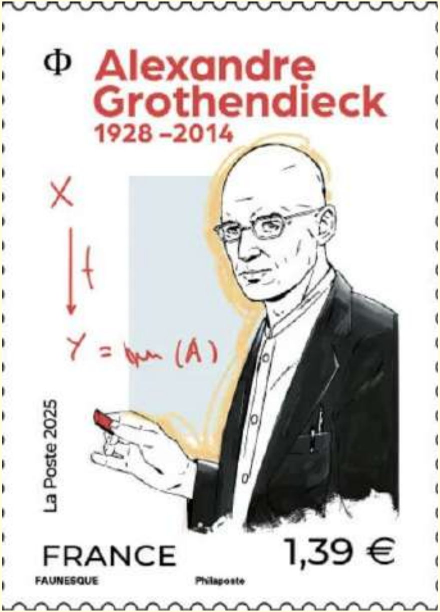
Thus

$$\mathrm{Hilb}^{dt+1-g}\mathbb{P}^3$$

is the parameter space for curves in $\mathbb{P}^3$ of degree d and arithmetic genus g.

Alexander Grothendieck showed the existence of Hilbert's schemes (not only for $\mathbb{P}^n$) in 1960.



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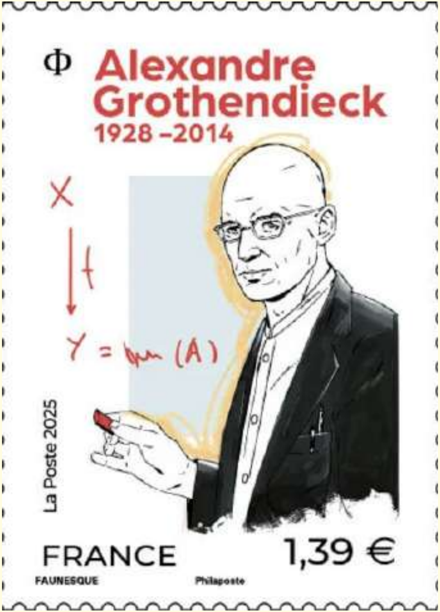
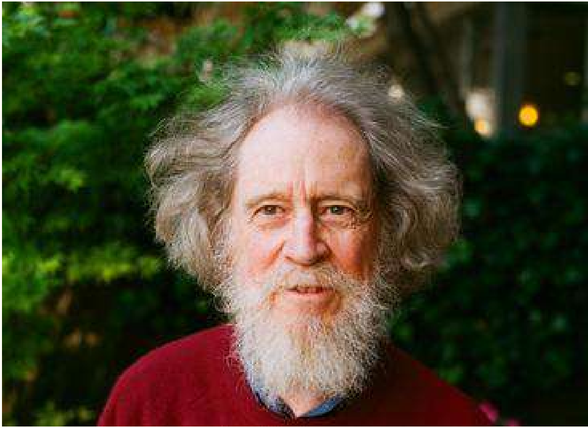


ALEXANDER GROTHENDIECK. Techniques de construction et théorèmes d'existence en géométrie algebrique, IV: Les schémas de Hillbert", Séminaire N. Bourbaki, 1961, exp. no 221, p. 249-276.

Robin Hartshorne showed that

$\mathrm{Hilb}^{p(t)}\mathbb{P}^n$

is connected in 1966.



ROBIN HARTSHORNE, Connectedness of the Hilbert scheme.

Publications mathématiques de l'I.H.É.S., tome 29 (1966), p. 5-48.

Some natural questions are:

Is the Hilbert scheme $\mathrm{Hilb}^{p(t)}\mathbb{P}^n$ irreducible?

How many irreducible components does the Hilbert scheme $\mathrm{Hilb}^{p(t)}\mathbb{P}^n$ have?

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Joe Harris and Ian Morrison "Moduli of Curves".

LAW (1.34) (MURPHY'S LAW FOR HILBERT SCHEMES) *There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.*

Some notation

Let us denote by

- x_0, x_1, x_2, x_3 the homogeneous coordinates in $\mathbb{P}^3$.
- $\mathcal{F}$ the vector space of linear forms
  (and as well the trivial vector bundle with fiber $\mathcal{F}$ over X).
- $\mathcal{F}_d = \mathrm{Sym}_d\mathcal{F}$ the d-th symmetric power of $\mathcal{F}$.
- $\mathbb{P}^N = \mathbb{P}(\mathcal{F}_d)$ the parameter space for surfaces of degree d in $\mathbb{P}^3$
  (with $N = \binom{3+d}{3} - 1$).

More notation...

- Let $\mathrm{G}_r(\mathcal{F}_d)$ denote the Grassmannian of r-dimensional linear subspaces of $\mathcal{F}_d$.
- In the Grassmannian of lines $\mathbb{G} := \mathrm{G}_2(\mathcal{F})$ we have the tautological sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

  where $rank\,\mathcal{A} = 2$ and its fiber over the line ℓ is the space of linear forms defining ℓ.
- Let $\mathcal{F}_d^\ell$ denote the subspace of $\mathcal{F}_d$ of forms of degree d that vanish on $\ell \in \mathbb{G}$.

Let us start with the following problem ("toy model")

Let $\mathbb{V}_d$ be the subvariety of $\mathbb{P}^N$ parametrizing surfaces of degree d containing a line. Determine the dimension and degree of $\mathbb{V}_d$ for $d \geq 4$.

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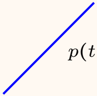
Note that lines are parametrized by:

$$\mathrm{Hilb}^{t+1}\mathbb{P}^3 \cong \mathbb{G}$$



$p(t) = t + 1 \qquad (d = 1,\ g = 0)$

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 & \mathbb{P}(\mathcal{F}_d^{|}) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
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Consequently,

$$\dim \mathbb{V}_d = \dim \mathbb{P}(\mathcal{F}_d^{|}) = \dim \mathbb{G} + N - (d+1) = N - (d-3).$$

So $deg(\mathbb{V}_d)$ is given by

$$deg(\mathbb{V}_d) = \int_{\mathbb{P}^N} c_1(\mathcal{O}_{\mathbb{P}^N}(1))^{4+(N-d-1)} \cap [\mathbb{V}_d],$$

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Therefore,

$$\boxed{deg(\mathbb{V}_d) = \int_{\mathbb{G}} c_4(\mathcal{Q}_d) \cap [\mathbb{G}].}$$

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$$c_k(\mathcal{O}_{\mathcal{Q}}(-1) \otimes S_{d-1}\mathcal{Q}) = \sum_{i=0}^{k} \binom{d-i}{k-i} c_i(S_{d-1}\mathcal{Q})(c_1(\mathcal{O}_{\mathcal{Q}}(-1)))^{k-i},$$

fornecem relações de recorrência entre a classe $c_j(S_d\mathcal{Q})$ e as classes $c_p(S_{d-1}\mathcal{Q})$, com $p = 1, \ldots, j$. A manipulação (enfadonha) dessas relações nos permite concluir que

$$\begin{aligned}
c_1(S_d\mathcal{Q}) &= \tbinom{d+1}{2} c_1(\mathcal{Q}) \\
c_2(S_d\mathcal{Q}) &= (3\tbinom{d+2}{4} - \tbinom{d+1}{3}) c_1(\mathcal{Q})^2 + \tbinom{d+2}{3} c_2(\mathcal{Q}) \\
c_3(S_d\mathcal{Q}) &= \tbinom{d+1}{2}\tbinom{d+1}{4} c_1(\mathcal{Q})^3 + 2d\tbinom{d+2}{4} c_1(\mathcal{Q}) c_2(\mathcal{Q}) \\
c_4(S_d\mathcal{Q}) &= (105\tbinom{d+3}{8} + 25\tbinom{d+2}{6} - \tbinom{d+1}{5}) c_1(\mathcal{Q})^4 + (10\tbinom{d+3}{6} - \tbinom{d+2}{5}) c_2(\mathcal{Q})^2 \\
&+ (105\tbinom{d+3}{7} - 5\tbinom{d+3}{6} + 23\tbinom{d+2}{5}) c_1(\mathcal{Q})^2 c_2(\mathcal{Q}).
\end{aligned} \tag{4.3}$$

JOSÉ MAIA. Geometria Enumerativa de Variedades projetivas Contendo Retas.
PhD Thesis in Mathematics - UFMG (2010). Advisor: Israel Vainsencher.

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So, putting classical intersection theory to work...

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$$deg(\mathbb{V}_d) = \binom{d+1}{4}(3d^4+6d^3+17d^2+22d+24)/24.$$

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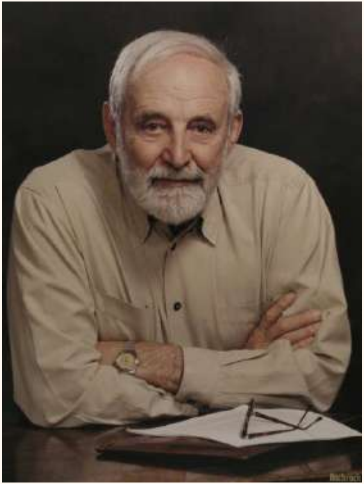

Raoul Bott, 1923-2005

Source: Harvard Gazette, September 2000

One way to get rid of the difficulty of find the Chow ring and solve a given enumerative problem is provided by the Bott's formula.

$$\int_X p(E) \cap [X] = \sum_{F \in X^T} \frac{p^T(E_{|F})}{c_d^T(\mathcal{T}_F X)}.$$

where $d = \dim X$.

Using Bott's formula to compute $deg(\mathbb{V}_d)$

Consider the group $\mathrm{T} = \mathbb{C} \setminus \{0\}$ and general weights $w_0, \dots, w_3 \in \mathbb{Z}$.

The usual T-action in $\mathcal{F}$ is given by

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In this case, the set of fixed points $\mathbb{G}^{\mathrm{T}} \subset \mathbb{G}$, is given by:

$$\mathbb{G}^T = \Big\{[x_0, x_1], [x_0, x_2], [x_0, x_3], [x_1, x_2], [x_1, x_3], [x_2, x_3]\Big\}.$$

It follows from Bott's formula that

$$deg(\mathbb{V}_d) = \int_{\mathbb{G}} c_4(\mathcal{Q}_d) \cap [\mathbb{G}] = \sum_{\ell \in \mathbb{G}^{\mathrm{T}}} \frac{c_4^{\mathrm{T}}(\mathcal{Q}_{d|\ell})}{c_4^{\mathrm{T}}(\mathcal{T}_\ell \mathbb{G})}$$

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- Compute the T-weights that appear in the T-equivariant decomposition of $\mathcal{E}_{|\ell}$.
- The class $c_k^{\mathrm{T}}(\mathcal{E}_{|\ell})$ is represented by the k-th elementary symmetric function in the T-weights of $\mathcal{E}_{|\ell}$.

Since $\mathcal{T}\mathbb{G} = \mathcal{A}^\vee \otimes \mathcal{Q}$. Then, for $\ell = [x_0, x_1] \in \mathbb{G}^{\mathrm{T}}$, we have that

$$\begin{aligned} \mathcal{T}_\ell\mathbb{G} &= [x_0, x_1]^\vee \otimes ([x_0, x_1, x_2, x_3]/[x_0, x_1]). \\ &= [x_0^\vee \otimes \overline{x_2}] \oplus [x_0^\vee \otimes \overline{x_3}] \oplus [x_1^\vee \otimes \overline{x_2}] \oplus [x_1^\vee \otimes \overline{x_3}] \end{aligned}$$

The eigen-space decomposition of $\mathcal{T}_\ell \mathbb{G}$

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$$\begin{array}{lcccc} vectors: & x_0^\vee \otimes \overline{x_2}, & x_0^\vee \otimes \overline{x_3}, & x_1^\vee \otimes \overline{x_2}, & x_1^\vee \otimes \overline{x_3}. \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ weights: & w_2 - w_0, & w_3 - w_0, & w_2 - w_1, & w_3 - w_1. \end{array}$$

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Thus $c_4^{\mathrm{T}}(\mathcal{T}_\ell\mathbb{G})$ is represented by

$$(w_2 - w_0) \cdot (w_3 - w_0) \cdot (w_2 - w_1) \cdot (w_3 - w_1)$$

in the T-equivariant Chow ring of the fixed point $\ell = [x_0, x_1]$.

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Thus $c_4^{\mathrm{T}}(\mathcal{Q}_{d|\ell})$ is represented by the coefficient of t^4 of

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in the T-equivariant Chow ring of the fixed point $\ell = [x_0, x_1]$.

d	$c_4^{\mathrm{T}}(\mathcal{Q}_{d\vert\ell})$
4	$8(3w_2^4 + 38w_3w_2^3 + 78w_3^2w_2^2 + 38w_3^3w_2 + 3w_3^4)$
5	$274w_2^4 + 2279w_3w_2^3 + 4269w_3^2w_2^2 + 2279w_3^3w_2 + 274w_3^4$
$\vdots$	$\vdots$

After some computations we get that

$$\sum_{\ell \in \mathbb{G}^{\mathrm{T}}} \frac{c_4^{\mathrm{T}}(\mathcal{Q}_{d|\ell})}{c_4^{\mathrm{T}}(\mathcal{T}_\ell \mathbb{G})} = \begin{cases} 320 & \text{if } d = 4 \\ 1990 & \text{if } d = 5 \\ 8680 & \text{if } d = 6 \\ 29960 & \text{if } d = 7 \\ 87402 & \text{if } d = 8 \\ \vdots & \vdots \end{cases}$$

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Question: For how many values of d do we need to perform the computation to determine $deg(\mathbb{V}_d)$?

In this case for exactly 9 values of d

Since, $deg(\mathbb{V}_d) = \binom{d+1}{4}(3d^4 + 6d^3 + 17d^2 + 22d + 24)/24$ has degree 8!

But, in general, we do not even know whether $deg(\mathbb{W}_d)$ is a polynomial, and in such a case, what its degree would be.

- Let W be a closed, irreducible subvariety of $\mathrm{Hilb}^{p(t)}\mathbb{P}^3 \;\; (deg(p(t)) = 1)$.
- Let $\mathbb{W}_d = \{[F] \in \mathbb{P}^N \mid [F]$ contains some member of $\mathrm{W}\}$.

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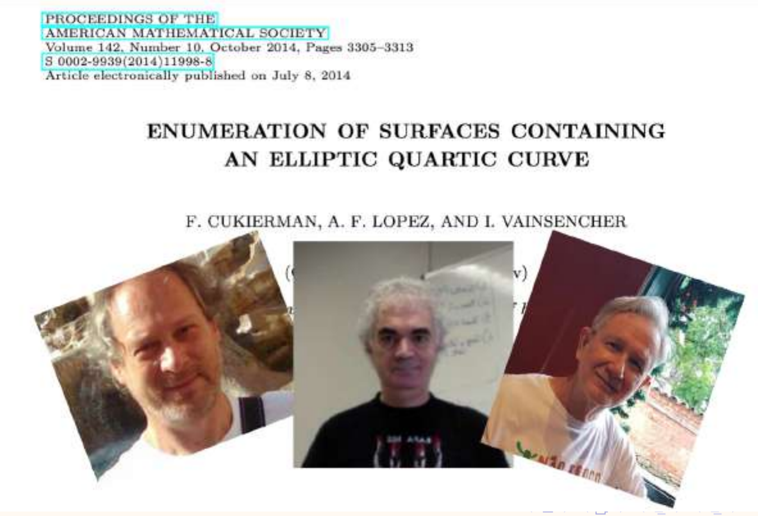
PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 10, October 2014, Pages 3305–3313
S 0002-9939(2014)11998-8
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ENUMERATION OF SURFACES CONTAINING
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$deg(\mathbb{W}_d)$ is a polynomial in d of degree $\leq 3\dim \mathrm{W}$, for all $d \gg 0$.

For example in the article

Journal of Pure and Applied Algebra 217 (2013) 1379–1394

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

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Enumeration of surfaces containing a curve of low degree

J.A.D. Maia [a], A.R. Silva [b], I. Vainsencher [c,*], F. Xavier [d]

[a] Departamento de Matemática, UFC, Fortaleza, CE, Brazil
[b] Departamento de Matemática, UFU, Uberlândia, MG, Brazil
[c] Departamento de Matemática, UFMG, Belo Horizonte, MG, Brazil
UFPB, João Pessoa, PB, Brazil




used the Bott's formula to compute

4.2.1. *Proposition*

Notations as above, the degree of the subvariety of $\mathbb{P}(\mathcal{F}_d)$ *formed by the surfaces containing two general lines is given by*

$$\binom{d}{3}\Big(45d^{13}+135d^{12}-345d^{11}+2655d^{10}-9115d^{9}+15657d^{8}-7371d^{7}$$
$$-157707d^{6}+716586d^{5}-1767492d^{4}+3318376d^{3}-4244928d^{2}+3723264d-5460480\Big)/2^{12}3^{3}5.$$

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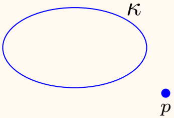

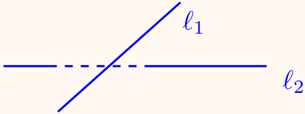

$$\mathbb{H}_1 \cong Bl_{\Sigma}(\mathbb{P}^3 \times \mathrm{Hilb}^{2t+1}\mathbb{P}^3)$$

where $\Sigma = \{(p,\kappa) \mid p \in \kappa\}$.

$$\dim \mathbb{H}_1 = 11$$

$$\mathbb{H}_2 \cong Bl_{\Delta}(\mathrm{Sym}^2\mathbb{G})$$

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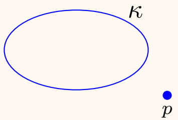

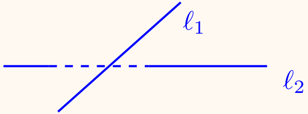

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$$\dim \mathbb{H}_2 = 8$$

In this paper, the authors also computed several other numbers: such as the degree in the case of surfaces containing three lines in general position.

Determine the degree of the subvariety $\mathbb{V}_d \subset \mathbb{P}(\mathcal{F}_d) = \mathbb{P}^N$ parameterizing surfaces of degree $d \geq 5$ containing a conic and two points varying on a fixed line.

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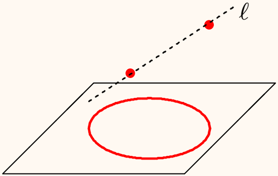
Note that, such a curve has Hilbert polynomial $p(t) = 2t + 3$.



For $d = 2$ and $g \in \{0, -1, -2, -3\}$

ALGEBRA AND NUMBER THEORY 6:4(2012)

msp

Detaching embedded points

Dawei Chen and Scott Nollet

S... osed subschemes $X \subset Y \subset \mathbb{P}^N$ differ at finite
...pecialization of X union isolated points? Our
...Y is a local complete intersection of codime...
...embedded point of Y is at most three. We
...n be weakened: the conclusion fails for
...n three, for local complete intersectio...
...or nonlocal complete intersections
...mine the irreducible components
...hmetic genus and show the sm...
...nember is a plane curve unic...

Example 4.6. For one-dimensional subschemes of degree 2 and high genus the irreducible components of $\mathrm{Hilb}^{2t+1-g}(\mathbb{P}^3)$ are as follows:

$\mathrm{Hilb}^{2t+3}\mathbb{P}^3 = \mathbb{H}_1 \cup \mathbb{H}_2 \cup \mathbb{H}_3$ has three irreducible components.

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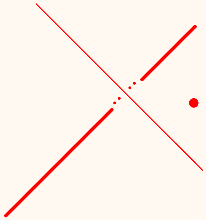
$\mathbb{H}_2$ has dimension 11 and its general point is the schematic union of a pair of skew lines and one point.



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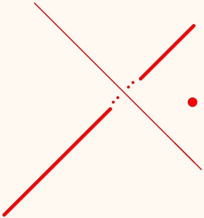
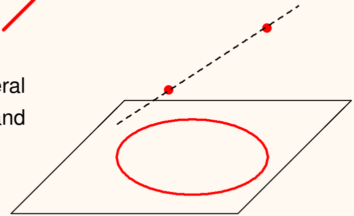
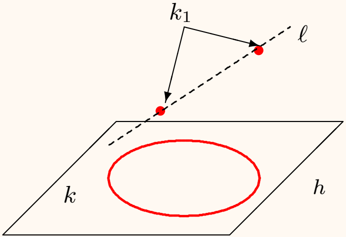
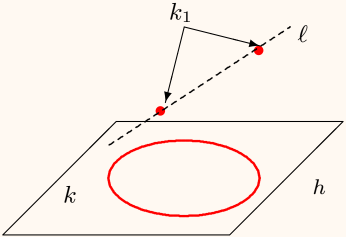


$\mathbb{H}_1$ has dimension 14 and its general point is the schematic union of a conic and two points.

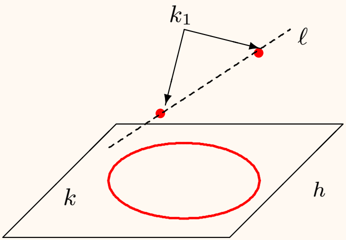


Using $\mathbb{H}_1$ to construct an appropriate parameter space



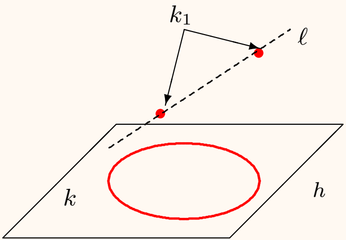


Assume that $\ell = [x_0, x_1] \in \mathbb{G}$. Thus



Assume that $\ell = [x_0, x_1] \in \mathbb{G}$. Thus

- $\kappa \sqcup \kappa_1 \longleftrightarrow \langle h, \kappa \rangle \cap \langle x_0, x_1, \kappa_1 \rangle =: \mathrm{I}.$



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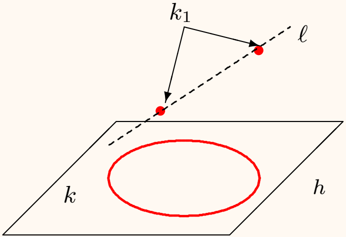

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- since such a curve has Hilbert polynomial $p(t) = 2t + 3$ then are expected $10 - 7 = 3$ generators for I_2 (as $\mathbb{C}$-vector space).
- as $\mathrm{I}_2 \supseteq [h \cdot x_0, h \cdot x_1]$ then a third generator q of I_2 will be defined module $[h \cdot x_0, h \cdot x_1]$.

Thus, we consider the bundle (here $\check{\mathbb{P}}^3 = \mathrm{G}_1(\mathcal{F})$)

$$\begin{array}{ccc} \mathbb{X} := \mathbb{P}\left(\mathcal{F}_2\Big/(\mathcal{O}_{\check{\mathbb{P}}^3}(-1)\otimes\mathcal{A})\right) & \longrightarrow & \check{\mathbb{P}}^3 \times \mathbb{G} \\ (b, \overline{q}) & \longmapsto & b = (h, \ell). \end{array}$$

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Note that, for $h \in \check{\mathbb{P}}^3$ and $\ell = [h_1, h_2] \in \mathbb{G}$, we have

$$\overline{q} \in \mathbb{P}(\mathcal{F}_2 / h \cdot \ell) \quad \text{where } h \cdot \ell = [hh_1, hh_2] \in \mathrm{G}_2(\mathcal{F}_2).$$

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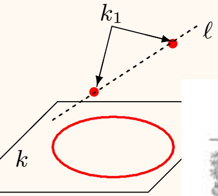
And, we obtain the rational map

$$\sigma : \mathbb{X} \cdots \longrightarrow \mathbb{H}_1 \quad (h, \ell, \overline{q}) \longmapsto \langle h, q \rangle \cap \langle h_1, h_2, q \rangle .$$



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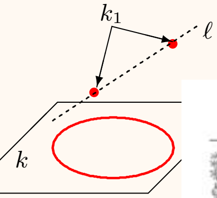
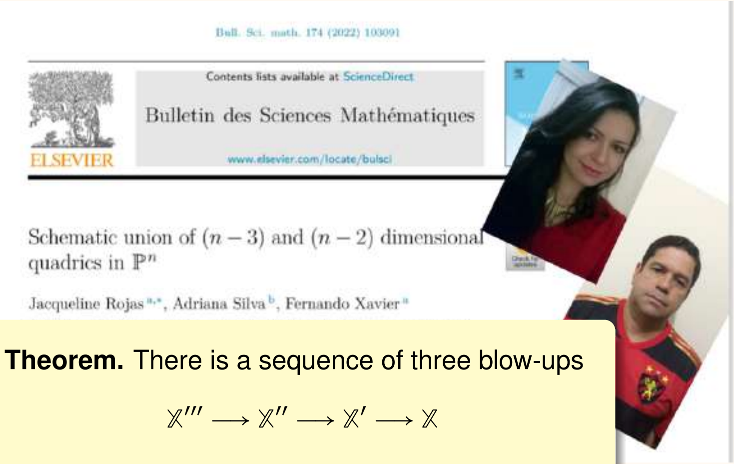
Schematic union of $(n-3)$ and $(n-2)$ dimensional quadrics in $\mathbb{P}^n$

Jacqueline Rojas [a,*], Adriana Silva [b], Fernando Xavier [a]

[a] CCEN, Departamento de Matemática, UFPB Cidade Universitária, 58051-900 João Pessoa, PB, Brazil
[b] Faculdade de Matemática, UFU, 38408-100 Uberlândia, MG, Brazil





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Schematic union of $(n-3)$ and $(n-2)$ dimensional quadrics in $\mathbb{P}^n$

Jacqueline Rojas[a,*], Adriana Silva[b], Fernando Xavier[a]

Theorem. There is a sequence of three blow-ups

$$\mathbb{X}''' \longrightarrow \mathbb{X}'' \longrightarrow \mathbb{X}' \longrightarrow \mathbb{X}$$

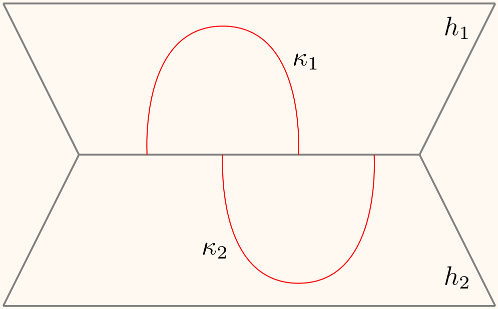
along explicit, smooth centers such that the map $\mathbb{X}''' \longrightarrow \mathbb{H}_1$ induced by σ is a morphism.

Theorem 15. *Notations as above, the degree of the subvariety $\mathbb{V}_d \subset \mathbb{P}(\mathcal{F}_d)$ parameterizing surfaces of degree d containing a conic and two points varying on a fixed line is given by*

$$\binom{d}{4}(207d^8 - 288d^7 + 498d^6 + 5068d^5 - 15693d^4 + 31732d^3 - 37332d^2 + 9280d - 47040) \cdot (d^2 - d + 8)(d^2 - d + 6) \Big/ 2^{10} \cdot 3^3 \cdot 5 \cdot 7.$$

The Hilbert scheme component of the schematic union of two conics

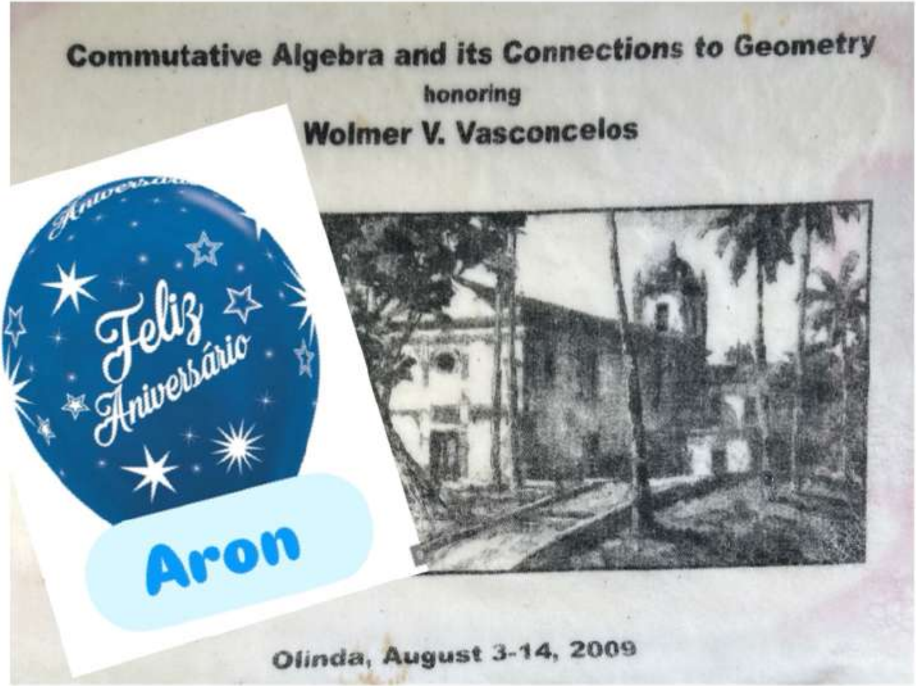
(joint work with A. Silva UFU and F. Xavier UFPB)



Theorem: The degree of the subvariety $\mathbb{V}_d \subset \mathbb{P}(\mathcal{F}_d)$ parameterizing surfaces of degree $d \geq 5$ containing a pair of conics with one of them varying in a fixed plane is given by

$$
\begin{aligned}
&(d-3)(d-2)(29601d^{24} - 277992d^{23} + 696696d^{22} + 8116394d^{21} \\
&-87456655d^{20} + 377297206d^{19} - 343879536d^{18} - 5655893100d^{17} \\
&+38897306019d^{16} - 123557368220d^{15} + 56150489824d^{14} \\
&+1770056663154d^{13} - 12287355080741d^{12} + 52412019401166d^{11} \\
&-165006827647656d^{10} + 395350639141616d^{9} - 710620777118928d^{8} \\
&+923754081008864d^{7} - 818227163469568d^{6} + 368390437933056d^{5} \\
&+628570428155904d^{4} - 3006776211513344d^{3} + 577480946171904 0d^{2} \\
&-4805170967347200d - 2338038743040000)/(2^{21} \cdot 3^{5} \cdot 5^{2} \cdot 7 \cdot 11 \cdot 13)
\end{aligned}
$$

Obrigada!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